

# EQUIDISTRIBUTION OF RANDOM WAVES ON SMALL BALLS

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**ABSTRACT.** Following [Ha2] by the first-named author, we continue our investigation of the equidistribution, at small scale, of random Laplacian eigenfunctions on a compact manifold  $\mathbb{M}$ . First we generalise the small scale expectation and variance results for random combinations of eigenfunctions to all compact manifolds. Then, assuming the same conditions as in [Ha2], i.e. the group of isometries acts transitively on  $\mathbb{M}$  and the multiplicity  $m_\lambda$  of eigenfrequency  $\lambda$  tends to infinity at least logarithmically as  $\lambda \rightarrow \infty$ , we improve the equidistribution of random eigenbases in [Ha2] to a smaller scale. In particular, on all  $n$ -dim spheres, we prove that the random eigenbasis is almost surely equidistributed up to the scale  $\lambda^{-1/2}$ .

## 1. INTRODUCTION

Studying the behaviour of random combinations of either plane waves or eigenfunctions has lately proved to be an exciting research area. It is conjectured, by Berry [B] in the 1970s, that eigenfunctions of chaotic systems such as billiards behave like random waves. On a compact manifold  $\mathbb{M}$ , the natural class of objects to randomise are eigenfunctions. That is, we study sums

$$\sum_{\lambda_j \in \Lambda} a_j e_j(x),$$

where  $e_j$  is an eigenfunction of the Laplacian on  $\mathbb{M}$ ,  $\Lambda \subset \mathbb{R}$ , and the coefficients  $a_j$  are prescribed in a random fashion. The obvious first question is how to pick the set  $\Lambda$ . Initially it may seem natural to fix an eigenspace  $E_\lambda$  and randomise only over the eigenfunctions with eigenvalue exactly  $\lambda$  (as done in [Ha2].) However, the multiplicity of this eigenvalue may be low (and in fact in chaotic cases such as when  $\mathbb{M}$  has negative curvature it is conjectured that the eigenvalues have very low multiplicity). Therefore, to capture the random behaviour, we allow ourselves to randomise over eigenfunctions whose eigenvalues sit in a spectral window (such randomisations were introduced in Zelditch [Z1]). That is, we consider the functions

$$u = \sum_{\lambda_j \in [\lambda-W, \lambda]} a_j e_j(x)$$

for spectral window widths  $1 \leq W \leq \lambda$ . Such functions are commonly referred to as “random waves”. We adopt this terminology and reserve the term “random eigenfunctions” for those combinations taken over a single eigenspace. We then ask about the expected behaviour of such functions as well as the variance in behaviour. In particular, we focus on small scale behaviour in this paper. We want to understand where random combinations of eigenfunctions equidistribute in small balls.

There are two parts to understanding this equidistribution. The first is to ascertain when

$$\mathbb{E} \left( \int_{B(x,r)} |u|^2 d\text{Vol} \right) \rightarrow \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} \text{ as } \lambda \rightarrow \infty. \quad (1.1)$$

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However, while the expectation value might equidistribute, it is still possible that the probability of non-equidistribution is quite high. To that end we also determine, for given  $r$ , whether

$$\text{Var} \left( \int_{B(x,r)} |u|^2 d\text{Vol} \right) = o(\text{Vol}(B, r)^2) \text{ as } \lambda \rightarrow \infty. \quad (1.2)$$

The variance estimate tells us whether we may expect that a typical eigenfunction equidistributes.

In addition to random waves, we also consider two different kinds of sets of random waves/eigenfunctions.

- Sequences of random waves as  $\lambda \rightarrow \infty$  in Corollary 1.5: For a sequence of  $\lambda_k \rightarrow \infty$  we choose a random wave for each  $k$  and consider the asymptotic behaviour of the sequence.
- Random eigenbases in Theorem 1.7: In the cases where the geometry of  $\mathbb{M}$  allows us to randomise over a single eigenspace we consider random basis sets for that eigenspace. Such bases can be thought of as arising from applying, to a fixed standard basis set, a random element of the unitary group with order equal to the multiplicity of the eigenspace. Examples on such manifolds include the spheres and the tori.

For the background of small scale equidistribution of eigenfunctions and its relations with quantum chaos, randomization, and other estimates of eigenfunctions, we refer to the recent works of Han [Ha1, Ha2], Hezari [He1, He2, He3], Hezari-Rivière [HR1, HR2], Lester-Rudnick [LR], Sogge [So1, So2], Zelditch [Z2], etc.

Let  $(\mathbb{M}, g)$  be an  $n$ -dim compact and smooth Riemannian manifold without boundary. Denote  $\Delta = \Delta_g$  the (positive) Laplace-Beltrami operator. Let  $\{e_j\}_{j=0}^\infty$  be an orthonormal basis of eigenfunctions (i.e. eigenbasis) of  $\Delta$  with eigenvalues  $\lambda_j^2$  (counting multiplicities), i.e.  $\Delta e_j = \lambda_j^2 e_j$ , where  $\lambda_j$  is called the eigenfrequency. Denote  $\text{Inj } \mathbb{M}$  the injectivity radius of  $\mathbb{M}$ . We assume, without loss of generality, that  $\text{Inj } \mathbb{M} \geq 1$ .

We formally define our probability space in a similar fashion to Zelditch [Z1].

**Definition 1.1.** Let  $N_W(\lambda)$  be the number of eigenfunctions (counted with multiplicity) in  $[\lambda - W, \lambda]$ . Then define

$$\mathcal{H}_W(\lambda) = \text{span}_{\lambda_j \in [\lambda - W, \lambda]} \{e_{\lambda_j}\}. \quad (1.3)$$

We introduce the following Gaussian probability measure on the space  $\mathcal{H}_W(\lambda)$ :

$$\gamma_W(\lambda) := \left( \frac{N_W(\lambda)}{\pi} \right)^{\frac{N_W(\lambda)}{2}} e^{-N_W(\lambda)|a|^2} da, \quad u_\lambda = \sum_{\lambda_j \in [\lambda - W, \lambda]} a_j e_j.$$

Here,  $da$  is the Lebesgue measure in  $\mathbb{R}^{N_W(\lambda)}$ .

Hence, the above Gaussian ensemble is equivalent to choosing  $u_\lambda \in \mathcal{H}_W(\lambda)$  at random from the unit sphere in  $\mathcal{H}_W(\lambda)$  with respect to the  $L^2$  inner product, that is, the expected value of  $\|u_\lambda\|_{L^2(\mathbb{M})}^2$  with respect to  $\gamma_W(\lambda)$  is 1. We will leverage this dual interpretation of randomisation a number of times. Our main theorem states

**Theorem 1.2** (Small scale equidistribution of random spectral clusters). *On a compact manifold  $\mathbb{M}$ , let  $1 \leq W \leq \lambda$ . Then*

- *for  $r > 0$ , the expected value with respect to the probability measure  $\gamma_W(\lambda)$*

$$\mathbb{E} \left( \int_{B(x,r)} |u_\lambda|^2 d\text{Vol} \right) = \frac{\text{Vol}(B(x, r))}{\text{Vol}(\mathbb{M})} (1 + O(W^{-1})) \quad (1.4)$$

*uniformly for all  $x \in \mathbb{M}$ ;*

- *for*

$$W \leq \lambda^{\frac{n-1}{2n-1}} \quad \text{and} \quad r^{-1} = o \left( W^{-\frac{1}{2(n-1)}} \lambda^{\frac{1}{2}} \right),$$

or

$$W \geq \lambda^{\frac{n-1}{2n-1}} \quad \text{and} \quad r^{-1} = o\left(W^{\frac{1}{2n}} \lambda^{\frac{n-1}{2n}}\right),$$

we have that the variance with respect to the probability measure  $\gamma_W(\lambda)$

$$\text{Var} \left( \int_{B(x,r)} |u_\lambda|^2 d\text{Vol} \right) = o(\text{Vol}(B(x,r))^2) \quad \text{as } \lambda \rightarrow \infty \quad (1.5)$$

uniformly for all  $x \in \mathbb{M}$ .

In the cases where the window width is growing  $\lambda$  we may therefore conclude that random spectral clusters become equidistributed as  $\lambda \rightarrow \infty$ .

**Corollary 1.3.** *Suppose that  $W = W(\lambda)$  and  $W^{-1} = o(1)$  as  $\lambda \rightarrow \infty$ . Then*

$$\mathbb{E} \left( \int_{B(x,r)} |u_\lambda|^2 d\text{Vol} \right) \rightarrow \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} \quad \text{as } \lambda \rightarrow \infty$$

uniformly for all  $x \in \mathbb{M}$ .

**Remark.** *In particular, if we choose  $W = \lambda$ , then  $\mathcal{H}_W(\lambda)$  is the cut-off frequency ensemble considered in Zelditch [Z1]. In this case, Theorem 1.2 states that the random waves with cut-off frequency are equidistributed up to the scale  $\lambda^{-1/2}$ .*

If we look at fixed (but large) window widths, i.e.  $W$  is independent of  $\lambda$ , we may only conclude that the  $L^2$  mass of  $u$  concentrated in the ball is proportional to the normalised volume of the ball, according to (1.4) in Theorem 1.2. However, with a geometric condition on the manifold  $\mathbb{M}$ , we may recover the result of Corollary 1.3. The relevant condition is, the set of the geodesic loop directions

$$\mathcal{L}_x := \{\xi \in S^*\mathbb{M} : G_t(x, \xi) = (x, \eta) \text{ for some } t > 0 \text{ and } \eta \in S_x^*\mathbb{M}\}$$

is of measure zero in  $S_x^*\mathbb{M}$  for all  $x \in \mathbb{M}$ . Here,  $S_x^*\mathbb{M}$  is the cosphere space of  $\mathbb{M}$  at  $x$  and  $S^*\mathbb{M}$  is the cosphere bundle of  $\mathbb{M}$ . Such pointwise aperiodic condition is called the “non self-focal” condition. Examples of manifolds satisfying the non-focal condition include the negatively curved manifolds (i.e. all sectional curvatures are negative everywhere.) Since manifolds with negative curvature are a key class of manifolds that we wish to understand using randomisation making such an assumption is not as restrictive as may first appear.

The above non self-focal condition is a natural dynamical condition to study the precise behavior of eigenfunctions restricted to a fixed-length spectrum window. See Section 2.1 for the background.

Concerning the small scale equidistribution of random eigenfunctions at asymptotically fixed frequency, we prove that

**Theorem 1.4.** *On a compact manifold  $\mathbb{M}$ , assume that the set of loop directions  $\mathcal{L}_x$  is of measure zero in  $S_x^*\mathbb{M}$  for all  $x \in \mathbb{M}$ . Then we have that for  $r > 0$ , the expected value with respect to the probability measure  $\gamma_W(\lambda)$*

$$\mathbb{E} \left( \int_{B(x,r)} |u_N|^2 d\text{Vol} \right) = \frac{\text{Vol}(B(x,r))}{\text{Vol}(\mathbb{M})} (1 + O(W^{-1})o(1)) \quad \text{as } \lambda \rightarrow \infty \quad (1.6)$$

uniformly for all  $x \in \mathbb{M}$ .

**Remark.** *In particular, if we choose  $W = 1$ , then  $\mathcal{H}_W(\lambda)$  is the asymptotically fixed frequency ensemble considered in Zelditch [Z1]. In this case, Theorem 1.4 together with (1.5) state that the random waves with asymptotically fixed frequency are equidistributed up to the scale  $\lambda^{-1/2}$ .*

We set the space of random sequences of eigenfunctions as  $\mathcal{H}_\infty := \prod_{N=0}^\infty \mathcal{H}_1(N)$  equipped with the probability measure  $\gamma_\infty := \prod_{N=1}^\infty \gamma_1(N)$ . Then we have that

**Corollary 1.5** (Small scale equidistribution of sequences of random waves). *On a compact manifold  $\mathbb{M}$ , assume that the set of loop directions  $\mathcal{L}_x$  is of measure zero in  $S_x^*\mathbb{M}$  for all  $x \in \mathbb{M}$ . Let  $1 \leq W \leq \lambda$ . Suppose that*

$$W \leq \lambda^{\frac{n-1}{2n-1}} \quad \text{and} \quad r^{-1} = o\left(W^{-\frac{1}{2(n-1)}} \lambda^{\frac{1}{2}}\right),$$

or

$$W \geq \lambda^{\frac{n-1}{2n-1}} \quad \text{and} \quad r^{-1} = o\left(W^{\frac{1}{2n}} \lambda^{\frac{n-1}{2n}}\right).$$

*Then almost surely with respect to  $\gamma_\infty$ , a random sequence of eigenfunctions  $\{u_j\} \in \mathcal{H}_\infty$  satisfies (1.9) uniformly for all  $x \in \mathbb{M}$ .*

In addition to these theorems that are valid for random spectral clusters on general manifolds, we are able to obtain some improvements random eigenfunctions on some special manifolds such as the sphere and the torus. In particular, we need to assume that  $\mathbb{M}$  satisfies

- (M1). the group of isometries acts transitively on  $\mathbb{M}$ ;
- (M2). the multiplicity  $m_\lambda$  of eigenfrequency  $\lambda$  satisfies

$$\liminf_{\lambda \rightarrow \infty} \frac{m_\lambda}{\log \lambda} > 0. \quad (1.7)$$

Let  $\mathcal{B}$  be the space of eigenbases with its natural probability measure.

The first named author, in [Ha2, Theorem 4], proved that

**Theorem 1.6** (Small scale equidistribution of random eigenbases). *Assume that  $\mathbb{M}$  satisfies (M1) and (M2). Let*

$$r_j = m_{\lambda_j}^{-\alpha} \quad \text{for } 0 \leq \alpha < \frac{1}{2n}. \quad (1.8)$$

*Then almost surely, a random eigenbasis  $\{u_j\} \in \mathcal{B}$  satisfies*

$$\int_{B(x, r_j)} |u_j|^2 d\text{Vol} = \frac{\text{Vol}(B(x, r_j))}{\text{Vol}(\mathbb{M})} + o(r_j^n) \quad \text{as } j \rightarrow \infty, \quad (1.9)$$

*uniformly for all  $x \in \mathbb{M}$ .*

In this paper we are able to improve the scale (1.8) in Theorem 1.6.

**Theorem 1.7** (Improved small scale equidistribution of random eigenbases). *Assume that  $\mathbb{M}$  satisfies (M1) and (M2). Let*

$$r_j = m_{\lambda_j}^{-\alpha} \quad \text{for } 0 \leq \alpha < \frac{1}{2(n-1)}. \quad (1.10)$$

*Then almost surely, a random eigenbasis  $\{u_j\} \in \mathcal{B}$  satisfies (1.9) uniformly for all  $x \in \mathbb{M}$ .*

**Remark.** *It is unknown to us now whether the scale  $r = \lambda^{-\frac{1}{2(n-1)}}$  in (1.10) can be improved further.*

From Theorem 1.7, we immediately derive the following results on the spheres and on the tori. These are the improvements of Han [Ha2, Corollaries 5 and 6], respectively. On  $\mathbb{S}^n$  ( $n \geq 2$ ),  $m_\lambda \gtrsim \lambda^{n-1}$ . So by Theorem 1.6,

**Corollary 1.8** (Small scale equidistribution of random eigenbases on the spheres). *On  $\mathbb{S}^n$  for  $n \geq 2$ , let*

$$r_j = \lambda_j^{-\rho} \quad \text{for } 0 \leq \rho < \frac{1}{2}.$$

*Then almost surely, a random eigenbasis  $\{u_j\} \in \mathcal{B}$  satisfies (1.9) uniformly for all  $x \in \mathbb{S}^n$ .*

Recall that the scale of equidistribution of random eigenbases on  $\mathbb{S}^n$  is  $\lambda^{-\rho}$  for  $0 \leq \rho < \frac{n-1}{2n}$  in Han [Ha2]. Hence, Corollary 1.8 improves the one in Han [Ha2] to a smaller scale  $\lambda^{-1/2}$  on  $\mathbb{S}^n$  of all dimensions. On  $\mathbb{T}^n$  ( $n \geq 5$ ),  $m_\lambda \gtrsim \lambda^{n-2}$ . So by Theorem 1.6,

**Corollary 1.9** (Small scale equidistribution of random eigenbases on the tori). *On  $\mathbb{T}^n$  for  $n \geq 5$ , let*

$$r_j = \lambda_j^{-\rho} \quad \text{for } 0 \leq \rho < \frac{n-2}{2(n-1)}.$$

*Then almost surely, a random eigenbasis  $\{u_j\} \in \mathcal{B}$  satisfies (1.9) uniformly for all  $x \in \mathbb{T}^n$ .*

Throughout this paper,  $A \lesssim B$  ( $A \gtrsim B$ ) means  $A \leq cB$  ( $A \geq cB$ ) for some constant  $c$  depending only on the manifold;  $A \approx B$  means  $A \lesssim B$  and  $B \lesssim A$ ; the constants  $c$  and  $C$  may vary from line to line.

## 2. PRELIMINARIES

In this section, we recall the spectral estimates of Laplacian and probabilistic estimates that are used in the proof of the theorems in this paper.

**2.1. Spectral estimates.** On a compact manifold  $\mathbb{M}$ , let  $\{e_j\}_{j=0}^\infty$  be an eigenbasis of  $\Delta$  with eigenvalues  $\lambda_j^2$ . The following Weyl law is proved by Hörmander [Ho].

**Theorem 2.1** (Pointwise Weyl law and Weyl asymptotics). *On a compact manifold  $\mathbb{M}$ ,*

$$\sum_{\lambda_j \leq \lambda} |e_j(x)|^2 = c_n \lambda^n + R(\lambda, x), \quad \text{where } R(\lambda, x) = O(\lambda^{n-1}) \text{ as } \lambda \rightarrow \infty, \quad (2.1)$$

*uniformly for all  $x \in \mathbb{M}$ . Here,  $c_n$  is a constant depending only on  $n$  (more precisely,  $c_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .) Integrating the above equation on  $\mathbb{M}$  with respect to  $x$ , we have the Weyl asymptotic of the distribution of eigenvalues. Let  $N(\lambda) := \#\{j : \lambda_j \leq \lambda\}$ . Then*

$$N(\lambda) = c_n \text{Vol}(\mathbb{M}) \lambda^n + R(\lambda), \quad \text{where } R(\lambda) = O(\lambda^{n-1}) \text{ as } \lambda \rightarrow \infty. \quad (2.2)$$

The remainder term estimate  $R(\lambda, x) = O(\lambda^{n-1})$  in (2.1) is sharp on the sphere  $\mathbb{S}^n$ . The  $\lambda^{n-1}$  growth rate is achieved at the poles of zonal harmonics on  $\mathbb{S}^n$ . (See Hörmander [Ho, Section 6].)

However, on some other manifolds than the spheres, the above estimates of  $R(\lambda, x)$  and  $R(\lambda)$  may be improved. Such improvements are related to the dynamical properties of the geodesic flow on  $\mathbb{M}$ . Let  $T^*\mathbb{M} = \{(x, \xi) : x \in \mathbb{M}, \xi \in T_x^*\mathbb{M}\}$  be the cotangent bundle of  $\mathbb{M}$ . Then the geodesic flow  $G_t$  is the Hamiltonian flow with Hamiltonian defined on  $T^*\mathbb{M}$  as  $H(x, \xi) = |\xi|_x^2$ , where  $|\cdot|_x$  is the induced metric on the cotangent space  $T_x^*\mathbb{M}$ . The geodesic flow  $G_t$  preserves the Liouville measure on  $T^*\mathbb{M}$ . Write the cosphere bundle  $S^*\mathbb{M} = \{(x, \xi) \in T^*\mathbb{M} : |\xi|_x = 1\}$ . Then  $G_t$  acts on  $S^*\mathbb{M}$  by homogeneity and leaves the induced Liouville measure on  $S^*\mathbb{M}$  invariant.

Denote the set of periodic geodesics on  $S^*\mathbb{M}$  as

$$\Pi = \{(x, \xi) \in S^*\mathbb{M} : G_t(x, \xi) = (x, \xi) \text{ for some } t > 0\}.$$

Duistermaat-Guillemin [DG] proved that

**Theorem 2.2** (Improved Weyl asymptotics). *Assume that the set of periodic geodesics  $\Pi$  is of Liouville measure zero in  $S^*\mathbb{M}$ . Then*

$$N(\lambda) = c_n \text{Vol}(\mathbb{M}) \lambda^n + R(\lambda), \quad \text{where } R(\lambda) = o(\lambda^{n-1}) \text{ as } \lambda \rightarrow \infty. \quad (2.3)$$

To get the improvement of pointwise Weyl law, we need a pointwise dynamical condition on the geodesics that is similar to the one in Theorem 2.2. A geodesic loop through  $x$  is a geodesic  $L(t)$  parametrized by arclength so that for some  $t_0 > 0$  such that  $L(0) = L(t_0) = x$ . Define the loop directions at  $x$  as

$$\mathcal{L}_x := \{\xi \in S^*\mathbb{M} : G_t(x, \xi) = (x, \eta) \text{ for some } t > 0 \text{ and } \eta \in S_x^*\mathbb{M}\}.$$

Sogge-Zelditch [SZ] proved that

**Theorem 2.3** (Improved pointwise Weyl estimate). *Assume that  $\mathcal{L}_x$  is of measure zero in  $S_x^*\mathbb{M}$  for all  $x \in \mathbb{M}$ . Then*

$$\sum_{\lambda_j \leq \lambda} |e_j(x)|^2 = c_n \lambda^n + R(\lambda, x), \quad \text{where } R(\lambda, x) = o(\lambda^{n-1}) \text{ as } \lambda \rightarrow \infty \quad (2.4)$$

*uniformly for all  $x \in \mathbb{M}$ .*

**Remark.**

- (1) *If  $\mathcal{L}_x$  is of measure zero on  $S_x^*\mathbb{M}$  for all  $x \in \mathbb{M}$ , then the set of periodic geodesics  $\Pi$  is of Liouville measure zero on  $S^*\mathbb{M}$ . Hence, one has that  $R(\lambda) = o(\lambda^{n-1})$  as  $\lambda \rightarrow \infty$  as a corollary of Theorem 2.3. (One can also instead integrate (2.4) on  $\mathbb{M}$  directly.)*
- (2) *In Sogge-Toth-Zelditch [STZ], they weakened the condition in Theorem 2.3. That is, suppose that the set of “recurrent loop directions” is of measure zero in  $S_x^*\mathbb{M}$  for all  $x \in \mathbb{M}$ , then (2.4) holds uniformly for all  $x \in \mathbb{M}$ . See also the earlier work by Safarov [Sa].*
- (3) *Sogge-Zelditch [SZ] and Sogge-Toth-Zelditch [STZ] addressed the problem: Determine the conditions of  $\mathbb{M}$  that ensure the maximal growth rate of eigenfunctions  $\|e_j\|_{L^\infty(\mathbb{M})} = \Omega(\lambda_j^{(n-1)/2})$  holds (or does not hold.) We refer to their work for more details on the relations between this problem and the estimates of the remainder  $R(\lambda, x)$ .*

**2.2. Probabilistic estimates.** Let  $\mathbb{S}^d \subset \mathbb{R}^{d+1}$  be the  $d$ -dim unit sphere endowed with the uniform probability measure  $\mu_d$ . Let

$$u = \sum_{j=1}^{d+1} a_j s_j, \quad \text{where } a = (a_1, \dots, a_{d+1}) \in \mathbb{S}^d \text{ and } s = (s_1, \dots, s_{d+1}) \in \mathbb{R}^{d+1}.$$

Notice that

$$|u| > t \quad \text{if and only if} \quad |\langle (a_1, \dots, a_{d+1}), (s_1(x), \dots, s_{d+1}(x)) \rangle_{\mathbb{R}^{d+1}}| > t.$$

We therefore have the following fact. See e.g. [BuLe, Appendix A] for an elementary proof.

**Lemma 2.4.**

$$\mu_d(|u| > t) = \begin{cases} \left(1 - \frac{t^2}{|s|^2}\right)^{\frac{d-1}{2}} & \text{if } 0 \leq t < |s|, \\ 0 & \text{if } t \geq |s|, \end{cases}$$

where  $|s|$  is the length of  $s = (s_1, \dots, s_{d+1}) \in \mathbb{R}^{d+1}$ .

The main driving force behind our control of the variance is the principle of concentration of measure. It is here that the high dimensionality of the probability spaces we consider comes into play. Concentration of measure requires that a random variable  $F(a)$  cannot take values away from its median too often. Exactly how close to the median depends on regularity properties of  $F$ . Let

$$\|F\|_{\text{Lip}} := \sup_{a \neq b} \frac{|F(a) - F(b)|}{\text{dist}(a, b)}$$

where  $\text{dist}(\cdot, \cdot)$  is the geodesic distance on  $\mathbb{S}^d$ . A number  $\mathcal{M}(F)$  is said to be a median value of  $F$  if

$$\mu_d(F \geq \mathcal{M}(F)) \geq \frac{1}{2} \quad \text{and} \quad \mu_d(F \leq \mathcal{M}(F)) \geq \frac{1}{2}.$$

Levy concentration of measures [Le, Theorem 2.3, (1.10), and (1.12)] then asserts that a Lipschitz function on  $\mathbb{S}^d$  is highly concentrated around its median value when  $d$  is large.

**Theorem 2.5** (Levy concentration of measures). *Consider a Lipschitz function  $F$  on  $\mathbb{S}^d$ . Then for any  $t > 0$ , we have*

$$\mu_d(|F - \mathcal{M}(F)| > t) \leq \exp \left( -\frac{(d-1)t^2}{2\|F\|_{\text{Lip}}^2} \right).$$

### 3. PROOF OF THEOREMS 1.2 AND 1.4

In this section, we prove the small scale equidistribution results of in Theorems 1.2 and 1.4. For  $a \in \mathbb{S}^{N_W(\lambda)-1}$ , let  $u_{\lambda,a} = \sum_{\lambda_j \in [\lambda-W, \lambda]} a_j e_j$ . Write

$$F_{x_0}(a) = \int_{B(x_0, r)} |u_{\lambda,a}(x)|^2 dx \quad \text{for } x_0 \in \mathbb{M}. \quad (3.1)$$

**Proposition 3.1.** *Suppose that  $F_{x_0}(a)$  is given by (3.1). Then*

$$\mathbb{E}(F_{x_0}) = \frac{\text{Vol}(B(x_0, r))}{\text{Vol}(\mathbb{M})} (1 + O(W^{-1})), \quad (3.2)$$

where the expectation is taken over  $\gamma_W(\lambda)$ .

*Proof.* Denote  $e_{\lambda,x} = |(e_1(x), \dots, e_{N_W(\lambda)}(x))|$ , i.e. the length of the vector  $(e_1(x), \dots, e_{N_W(\lambda)}(x)) \in \mathbb{R}^{N_W(\lambda)}$ . Then by the pointwise Weyl law in Theorem 2.1, we have that

$$e_{W,\lambda}^2(x) = c_n(\lambda^n - (\lambda - W)^n) + O(\lambda^{n-1}) \quad \text{uniformly for all } x \in \mathbb{M}.$$

Therefore we have the local asymptotic

$$e_{W,\lambda}^2(x) = c_n n W \lambda^{n-1} + O(\lambda^{n-1}),$$

and integration over  $\mathbb{M}$  gives

$$N_W(\lambda) = c_n n W \lambda^{n-1} \text{Vol}(\mathbb{M}) + O(\lambda^{n-1}).$$

We compute the expected value of  $F_{x_0}$  with respect to  $\gamma_W(\lambda)$ , equivalently, the expected value of the square of the  $L^2$  mass of a random wave  $u_\lambda$  from the unit sphere in  $\mathcal{H}_W(\lambda)$ . By Lemma 2.4, we have that for  $r > 0$ ,

$$\begin{aligned} \mathbb{E}(F_{x_0}) &= \int_{\mathbb{S}^{N_W(\lambda)-1}} \int_{B(x_0, r)} |u_{\lambda,a}(x)|^2 dx d\mu_{N_W(\lambda)-1} \\ &= \int_{B(x_0, r)} \int_{\mathbb{S}^{N_W(\lambda)-1}} |u_{\lambda,a}(x)|^2 d\mu_{N_W(\lambda)-1} dx \\ &= \int_{B(x_0, r)} 2 \int_0^\infty t \mu_{N_W(\lambda)-1}(|u_{\lambda,a}(x)| > t) dt dx \\ &= \int_{B(x_0, r)} 2 \int_0^{e_{W,\lambda}(x)} t \left( 1 - \frac{t^2}{e_{W,\lambda}^2(x)} \right)^{N_W(\lambda)-1} dt dx \\ &= \frac{1}{N_W(\lambda)} \int_{B(x_0, r)} e_{W,\lambda}^2(x) dx \\ &= \frac{\text{Vol}(B(x_0, r))}{\text{Vol}(\mathbb{M})} (1 + O(W^{-1})). \end{aligned} \quad (3.3)$$

□

To study the variance, a key ingredient is the control on the worst decay rate of  $\|u\|_{L^2(B(x,r))}$  in a small ball  $B(x, r)$  when  $r \rightarrow 0$  as  $\lambda \rightarrow \infty$ . This estimate will directly allow us to control Lipschitz norm of  $\mathcal{F}_{x_0}$ , which in turn controls the level of concentration. Sogge [So2, (4.1)] proved the following estimate of  $L^2$  mass of eigenfunctions and spectral clusters of width 1 on small balls.

**Lemma 3.2.** *On a compact manifold  $\mathbb{M}$ , let  $u = \sum_{\lambda_j \in [\lambda-1, \lambda]} c_j e_j$ . Then for all  $x \in \mathbb{M}$  and  $\lambda^{-1} \leq r \leq \text{Inj } \mathbb{M}$ , we have that*

$$\int_{B(x, r)} |u|^2 d\text{Vol} \leq cr \|u\|_{L^2(\mathbb{M})}^2,$$

where  $c$  depending only on  $\mathbb{M}$ .

Lemma 3.2 is of course an improvement on the trivial estimate  $\int_{B(x, r)} |u|^2 d\text{Vol} \leq 1$ . In fact, it is already sharp on  $\mathbb{S}^n$ , as the equation is achieved by the zonal harmonics on balls centered at its poles on  $\mathbb{S}^n$ . See Sogge [So2, Section 4] for more discussion.

Since we are considering window widths that may be much larger than the one in Lemma 3.2, we need an equivalent statement for

$$u = \sum_{\lambda_j \in [\lambda-W, \lambda]} a_j e_j$$

Fortunately we are able to use Lemma 3.2 to very easily obtain the necessary estimates.

**Lemma 3.3.** *Suppose that*

$$u = \sum_{\lambda_j \in [\lambda-W, \lambda]} a_j e_j.$$

Then

$$\int_{B(x, r)} |u|^2 d\text{Vol} \leq \begin{cases} cWr \|u\|_{L^2(\mathbb{M})}^2 & \text{if } \lambda^{-1} \leq r \leq W^{-1}, \\ \|u\|_{L^2(\mathbb{M})}^2 & \text{if } W^{-1} \leq r \leq \text{Inj } \mathbb{M}. \end{cases} \quad (3.4)$$

*Proof.* Clearly the estimate of (3.4) is trivial when  $r \geq W^{-1}$ , so we may assume  $\lambda^{-1} \leq r \leq W^{-1}$ . Also if  $W \in [\lambda/2, \lambda]$  the estimate of (3.4) is trivial so we may assume  $W \leq \lambda/2$ . We write

$$u = \sum_{k=0}^{W-1} u_k, \quad \text{in which} \quad u_k = \sum_{\lambda_j \in [\lambda-W+k, \lambda-W+k+1]} a_j e_j.$$

Note that each  $u_k$  is a fixed window spectral cluster at frequency  $\mu_k = \lambda - W + k + 1 > \lambda/2$  so we may apply Lemma 3.2 to each of the  $u_k$  separately. Now

$$\int_{B(x_0, r)} |u|^2 d\text{Vol} = \sum_{m=0}^{W-1} \sum_{k=0}^{W-1} \int_{B(x_0, r)} u_k(x) \bar{u}_{(k+m)_{W-1}}(x) d\text{Vol}$$

where

$$(k+m)_{W-1} = k+m \pmod{W-1}.$$

So by applying Lemma 3.2 and the Cauchy-Schwartz inequality, we have that

$$\begin{aligned} \int_{B(x_0, r)} |u|^2 d\text{Vol} &\lesssim r \sum_{m=0}^{W-1} \sum_{k=0}^{W-1} \|u_k\|_{L^2(\mathbb{M})} \|u_{(k+m)_{W-1}}\|_{L^2(\mathbb{M})} \\ &\lesssim r \sum_{m=0}^{W-1} \left( \sum_{k=0}^{W-1} \|u_k\|_{L^2(\mathbb{M})}^2 \right)^{1/2} \left( \sum_{k=0}^{W-1} \|u_{(k+m)_{W-1}}\|_{L^2(\mathbb{M})}^2 \right)^{1/2} \\ &\lesssim rW \|u\|_{L^2(\mathbb{M})}^2. \end{aligned}$$

□

**Remark 3.4.** *It turns out that these simple estimates are sharp. Spectral clusters  $u$  of window width  $W$  in Lemma 3.3 can be thought of as approximate eigenfunctions with  $L^2$  error no greater than  $W\lambda$ . That is*

$$\|(\Delta - \lambda^2)u\|_{L^2(\mathbb{M})} \lesssim W\lambda \|u\|_{L^2(\mathbb{M})}.$$



Such functions can be localised so that all their  $L^2$  is located in one  $W^{-1}$  size ball. See e.g. Tacy [T].

We are now in a position to control the variance of  $F_{x_0}$ .

**Proposition 3.5.** *Suppose  $F_{x_0}$  is given by (3.1) and*

$$r^{-1} = o\left(W^{-\frac{1}{2(n-1)}} \lambda^{\frac{1}{2}}\right).$$

Then

$$\text{Var}(F_{x_0}) = o(\text{Vol}(B(x_0, r))^2).$$

*Proof.* To compute the variance of  $F_{x_0}$ , observe that

$$\begin{aligned} \text{Var}(F_{x_0}) &= \int_{\mathbb{S}^{N_W(\lambda)-1}} |F_{x_0}(c) - \mathbb{E}(F_{x_0}(c))|^2 d\mu_{N_W(\lambda)-1} \\ &= \|F_{x_0} - \mathbb{E}(F_{x_0})\|_{L^2(\mathbb{S}^{N_W(\lambda)-1})}^2 \\ &\leq \left( \|F_{x_0} - \mathcal{M}(F_{x_0})\|_{L^2(\mathbb{S}^{N_W(\lambda)-1})} + \|\mathbb{E}(F_{x_0}) - \mathcal{M}(F_{x_0})\|_{L^2(\mathbb{S}^{N_W(\lambda)-1})} \right)^2 \\ &\leq 2\|F_{x_0} - \mathcal{M}(F_{x_0})\|_{L^2(\mathbb{S}^{N_W(\lambda)-1})}^2 + 2\|\mathbb{E}(F_{x_0}) - \mathcal{M}(F_{x_0})\|^2, \end{aligned}$$

where  $\mathcal{M}(F_{x_0}(u_\lambda))$  is a median value of  $F_{x_0}$  with respect to  $\gamma_W(\lambda)$ .

To estimate the median value of  $F_{x_0}$ , we use Levy concentration of measures in Theorem 2.5, that is,  $F_{x_0}$  concentrates at an exponential rate depending on  $\|F_{x_0}\|_{\text{Lip}}$  around its median value.

Given  $u, v \in \mathcal{H}_W(\lambda)$ , let

$$u = \sum_{j=1}^{N_W(\lambda)} a_j e_j \quad \text{and} \quad v = \sum_{j=1}^{N_W(\lambda)} b_j e_j,$$

where  $a = (a_1, \dots, a_{N_W(\lambda)})$  and  $b = (b_1, \dots, b_{N_W(\lambda)})$  are in  $\mathbb{S}^{N_W(\lambda)-1}$ . Thinking of  $F_{x_0}(a)$  ( $F_{x_0}(b)$ ) as the square of the  $L^2$  mass of the function  $u$  ( $v$ ) we have that

$$\begin{aligned} |F_{x_0}(a) - F_{x_0}(b)| &\leq \int_{B(x_0, r)} ||u(x)|^2 - |v(x)|^2| dx \\ &= \int_{B(x_0, r)} |(|u(x)| - |v(x)|)(|u(x)| + |v(x)|)| dx \\ &\leq \left( \int_{B(x_0, r)} |u(x) - v(x)|^2 dx \right)^{\frac{1}{2}} \left( \int_{B(x_0, r)} (|u(x)| + |v(x)|)^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

Now if  $r \leq W^{-1}$  by applying Lemma 3.3 we obtain that

$$\|u - v\|_{L^2(B(x_0, r))} \leq Cr^{1/2} W^{1/2} \|u - v\|_{L^2(\mathbb{M})},$$

and

$$\|u + v\|_{L^2(B(x_0, r))} \leq Cr^{1/2} W^{1/2} \|u - v\|_{L^2(\mathbb{M})};$$

and if  $r > W^{-1}$ , we obtain that

$$\|u - v\|_{L^2(B(x_0, r))} \leq C \|u - v\|_{L^2(\mathbb{M})},$$

and

$$\|u + v\|_{L^2(B(x_0, r))} \leq C \|u - v\|_{L^2(\mathbb{M})}.$$

Since

$$u(x) - v(x) = \sum_{j=1}^{N_W(\lambda)} (a_j - b_j) e_j(x)$$

and  $e_j$  are orthonormal we have that

$$\|u - v\|_{L^2(\mathbb{M})} \approx \text{dist}(a, b).$$

So

$$\|F_{x_0}\|_{\text{Lip}} \leq \begin{cases} crW & \text{if } \lambda^{-1} \leq r \leq W^{-1}, \\ 1 & \text{if } W^{-1} \leq r \leq \text{Inj}(\mathbb{M}). \end{cases}$$

Therefore using the Levy concentration of measure we have that

$$\begin{aligned} \|F_{x_0} - \mathcal{M}(F_{x_0})\|_{L^2(\mathbb{S}^{N_W(\lambda)-1})}^2 &= \int_{\mathbb{S}^{N_W(\lambda)-1}} |F_{x_0}(a) - \mathcal{M}(F_{x_0})|^2 d\mu_{N_W(\lambda)-1} \\ &= 2 \int_0^\infty t \mu_{N_W(\lambda)-1}(|F_{x_0}(a) - \mathcal{M}(F_{x_0})| > t) dt \\ &\leq 2 \int_0^\infty t \exp\left(-\frac{(N_W(\lambda) - 2)t^2}{2\|F_{x_0}\|_{\text{Lip}}^2}\right) dt \\ &\leq \frac{\|F_{x_0}\|_{\text{Lip}}^2}{N_W(\lambda)}. \end{aligned}$$

So

$$\|F_{x_0}(u_\lambda) - \mathcal{M}(F_{x_0}(u_\lambda))\|_{L^2(\mathbb{S}^{N_W(\lambda)-1})}^2 \lesssim \begin{cases} \frac{cr^2W^2}{N_W(\lambda)} & \text{if } \lambda^{-1} \leq r \leq W^{-1}, \\ \frac{c}{N_W(\lambda)} & \text{if } W^{-1} \leq r \leq \text{Inj}(\mathbb{M}). \end{cases} \quad (3.5)$$

Also,

$$\begin{aligned} |\mathbb{E}(F_{x_0}) - \mathcal{M}(F_{x_0})| &= \left| \|F_{x_0}\|_{L^1(\mathbb{S}^{N_W(\lambda)-1})} - \|\mathcal{M}(F_{x_0})\|_{L^1(\mathbb{S}^{N_W(\lambda)-1})} \right| \\ &\leq \|F_{x_0}(u_\lambda) - \mathcal{M}(F_{x_0})\|_{L^1(\mathbb{S}^{N_W(\lambda)-1})} \\ &= \int_0^\infty \mu_{N_W(\lambda)-1}(|F_{x_0}(a) - \mathcal{M}(F_{x_0}(u_\lambda))| > t) dt \\ &\leq \int_0^\infty \exp\left(-\frac{(N_W(\lambda) - 2)t^2}{2\|F_{x_0}\|_{\text{Lip}}^2}\right) dt \\ &\leq \frac{\|F_{x_0}\|_{\text{Lip}}}{\sqrt{N_W(\lambda)}}. \end{aligned}$$

Hence,

$$|\mathbb{E}(F_{x_0}) - \mathcal{M}(F_{x_0})| \leq \begin{cases} \frac{crW}{\sqrt{N_W(\lambda)}} & \text{if } \lambda^{-1} \leq r \leq W^{-1} \\ \frac{c}{\sqrt{N_W(\lambda)}} & \text{if } W^{-1} \leq r \leq \text{Inj}(\mathbb{M}). \end{cases} \quad (3.6)$$

Putting (3.5) and (3.6) together we obtain

$$\text{Var}(F_{x_0}) \leq \begin{cases} \frac{cr^2W^2}{N_W(\lambda)} & \text{if } \lambda^{-1} \leq r \leq W^{-1} \\ \frac{c}{N_W(\lambda)} & \text{if } W^{-1} \leq r \leq \text{Inj}(\mathbb{M}). \end{cases} \quad (3.7)$$

From the Weyl asymptotics in Theorem 2.1, we have that  $N_W(\lambda) = c_n W \lambda^{n-1} \text{Vol}(\mathbb{M}) + O(\lambda^{n-1})$ . So if either

$$W \leq \lambda^{\frac{n-1}{2n-1}} \quad \text{and} \quad r^{-1} = o\left(W^{-\frac{1}{2(n-1)}} \lambda^{\frac{1}{2}}\right),$$

or

$$W \geq \lambda^{\frac{n-1}{2n-1}} \quad \text{and} \quad r^{-1} = o\left(W^{\frac{1}{2n}} \lambda^{\frac{n-1}{2n}}\right),$$

we obtain that

$$\text{Var}(F_{x_0}) = o(\text{Vol}(B(x_0, r))^2) = o(r^{2n}).$$

□

To prove Theorem 1.4 under the non-focal condition given in Section 2, we argue similarly as the above. One only needs to notice that, if every point on  $\mathbb{M}$  is non-focal, then the pointwise Weyl Law in Theorem 2.3 asserts that

$$\sum_{\lambda_j \leq \lambda} |e_j(x)|^2 = c_n \lambda^n + o(\lambda^{n-1}) \text{ as } \lambda \rightarrow \infty$$

uniformly for all  $x \in \mathbb{M}$ . So in a spectral window  $[\lambda - W, \lambda]$  with length  $W$  fixed, we have that as in Proposition 3.1,

$$e_{W,\lambda}^2(x) = c_n n W \lambda^{n-1} + o(\lambda^{n-1}),$$

and integration over  $\mathbb{M}$  gives

$$N_W(\lambda) = c_n n W \lambda^{n-1} \text{Vol}(\mathbb{M}) + o(\lambda^{n-1}).$$

We can then estimate the expectation for a random eigenfunction  $e_{\lambda,a}$  in  $\mathcal{H}_W(\lambda)$  in the same way as in Proposition 3.1.

#### 4. PROOF OF THEOREM 1.7

In this section, we improve the scale of equidistribution of random eigenbases in Han [Ha2]. We begin by recalling the notations of random eigenbases used in Han [Ha2]. On a compact manifold  $\mathbb{M}$ , write  $L^2(\mathbb{M}) = \bigoplus_{k=0}^{\infty} E_k$ , where  $E_k$  is the eigenspace of  $\Delta$  with eigenvalue  $\lambda_k^2$ . Denote  $m_k := m_{\lambda_k} = \dim(E_k)$  as the multiplicity of  $\lambda_k$ .

Then any eigenfunction  $u \in E_k$  can be written as

$$u(x) = \sum_{i=1}^{m_k} a_i e_{i,k}(x), \quad \text{where } (a_1, \dots, a_{m_k}) \in \mathbb{S}_{\mathbb{C}}^{m_k-1}.$$

Here, we identify  $\mathbb{S}_{\mathbb{C}}^{m_k-1}$  with probability measure  $P_k$  by  $\mathbb{S}^{2m_k-1}$  with probability measure  $\mu_{2m_k-1}$ .

The space of eigenbases in  $E_k$  can be identified as  $\mathcal{B}_k \cong \mathbb{U}(m_k)$ . Here,  $\mathbb{U}(m_k)$  is the unitary group on  $\mathbb{C}^{m_k}$  endowed with the probability measure as the Haar measure  $\nu_k$ . Hence, the space of eigenbases  $\mathcal{B}$  can then be identified as  $\mathcal{B} \cong \times_{k=0}^{\infty} \mathbb{U}(m_k)$  endowed with the product probability measure  $\nu := \bigotimes_{k=0}^{\infty} \nu_k$ .

**Remark.** One can also consider the real randomization here. That is, any  $L^2$ -normalized eigenfunction in  $E_k$  is written as

$$u(x) = \sum_{i=1}^{m_k} u_i e_{i,k}(x), \quad \text{where } (u_1, \dots, u_{m_k}) \in \mathbb{S}_{\mathbb{R}}^{m_k-1}.$$

The space of eigenbases  $\tilde{\mathcal{B}}$  can be identified by

$$\tilde{\mathcal{B}} \cong \times_{k=0}^{\infty} \mathbb{O}(m_k) \quad \text{with product probability measure } \tilde{\nu} := \bigotimes_{k=0}^{\infty} \tilde{\nu}_k,$$

where  $\tilde{\nu}_k$  is the Haar measure on the orthogonal group  $\mathbb{O}(m_k)$  on  $\mathbb{R}^{m_k}$ .

As indicated in Han [Ha2], Theorems 1.6 and 1.7 are valid in the setting of real randomization, and the proofs are similar.

For a random eigenbasis  $\{u_{i,k}\} \in \mathcal{B}$ , we consider the expected behaviour on a small ball about some fixed  $x_0$ .

From (M1), the probability distribution of the length  $|u(x)|$  of the vector  $u(x)$  is independent of  $x$ . (See [Ha2, Lemma 8].) It is this key fact that allows the expectation to be calculated without relying on pointwise Weyl laws (and their attendant remainder terms). In Han [Ha2, Section 3.1] it was shown that

$$\mathbb{E}(u_{i,k}) \rightarrow \frac{\text{Vol}(B(x_0, r_k))}{\text{Vol}(\mathbb{M})} \text{ as } k \rightarrow \infty. \quad (4.1)$$

We then estimate the median value of  $F_{x_0}$ , we use Levy concentration of measures in Theorem 2.5 similarly as in Section 3. Since we are now dealing with exact eigenfunctions we may directly apply Lemma 3.2 on small balls to obtain.

$$\|F_{x_0}(u)\|_{\text{Lip}} \leq cr_k. \quad (4.2)$$

Thus by Theorem 2.5, we have for  $r_k = m_k^{-\alpha}$ ,

$$\begin{aligned} |\mathbb{E}(F_{x_0}) - \mathcal{M}(F_{x_0})| &= \left| \|F_{x_0}\|_{L^1(\mathbb{S}_{\mathbb{C}}^{m_k-1})} - \|\mathcal{M}(F_{x_0})\|_{L^1(\mathbb{S}_{\mathbb{C}}^{m_k-1})} \right| \\ &\leq \|F_{x_0} - \mathcal{M}(F_{x_0})\|_{L^1(\mathbb{S}_{\mathbb{C}}^{m_k-1})} \\ &= \int_0^\infty P_k(|F_{x_0}(a) - \mathcal{M}(F_{x_0})| > t) dt \\ &\leq \int_0^\infty \exp\left(-\frac{(m_k-1)t^2}{\|F_{x_0}\|_{\text{Lip}}^2}\right) dt \\ &\leq cm_k^{-\frac{1}{2}-\alpha}. \end{aligned}$$

Therefore, when

$$0 \leq \alpha < \frac{1}{2(n-1)},$$

we have

$$|\mathbb{E}(F_{x_0}) - \mathcal{M}(F_{x_0})| = O(m_k^{-1/2-\alpha}) = o(m_k^{-\alpha n}) = o(r_k^n).$$

Hence, seeing (4.1),

$$\mathcal{M}(F_{x_0}) = \frac{\text{Vol}(B(x_0, r_k))}{\text{Vol}(\mathbb{M})} + o(r_k^n). \quad (4.3)$$

Once we have the above estimate of the median value of  $F_{x_0}$ , we then follow Han [Ha2] to use the fact that  $F_{x_0}$  concentrates at an exponential rate around its median. Hence, we prove (1.9) at this fixed point  $x_0$  on the manifold. To complete the proof for all points uniformly, we use a covering argument that is similar to the one in Han [Ha2], and we here omit the details.

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